THE EFFECT OF STAGGERED WORK HOURS ON WORK START TIMES AND BOTTLENECK CONGESTION

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ABSTRACT

In this study, we examine the equilibrium patterns of work start times and bottleneck congestion by extending the Henderson (1981)'s model to incorporate the bottleneck congestion. Analyzing the model, we show that 1) the extended model belongs to the class of potential game, 2) equilibrium of the model is generally non-unique, and 3) in the case that each firm chooses the work start time from two alternatives, the stable equilibrium patterns of work start times and bottleneck congestion is uniquely determined.

INTRODUCTION

Traffic congestion in cities is caused by concentration of travel demand around the work start time. This results from the fact that most firms in a central business district (CBD) adopt fixed work schedule and workers begin their work at the same time. The introduction of staggered work hours is one of the transportation demand management (TDM) measures for alleviating peak congestion. This measure is widely recognized as a means to significantly reduce the level of congestion, but hardly implemented. Because staggered work hours may lead to decline in mutual communication between firms, and there is a possibility that total productivity in the city decreases. That is, there is a trade-off between peak congestion and productivity effect. In fact, social experiments of TDM measures in Japan report strong reluctance of firms to join the staggered work hours and explanation of its negative effect on business efficiency (see, Yoshimura and Okumura, 2001).

Henderson (1981) provides an important benchmark for the study of staggered work hours. He develops a model of work start time choice which incorporates the productivity effect, and analyzes the equilibrium and optimum distribution of work start times and congestion patterns. Arnott et al. (2005) extends Henderson's model to treat firm heterogeneity. However, in their study, traffic congestion is described by a flow congestion model, which means that their formulation is not appropriate for dealing with peak (queuing) congestion. Moreover, they focused only on the case that work hours are completely staggered, although it is reported that work start times are clustered at several points, such as 8:30, 9:00, and 9:30. Sato and Akamatsu (2006) develops a simultaneous equilibrium model of work start time and departure time choices with bottleneck congestion (Vickrey, 1969; Arnott et al., 1993). They analyze equilibrium solutions with/without staggered work hours, and show that there can be multiple equilibria. However, Sato and Akamatsu (2006) does not investigate the stability of equilibria and optimum distribution of work start times. In order to examine the effect and feasibility of staggered work hours, it is important to analyze stability of equilibria and optimum of the model.

In this study, we extend the Henderson (1981)'s model to incorporate the bottleneck congestion and discrete work start times, and examine equilibrium patterns of work start times and bottleneck congestion. To achieve the purpose, we first show that our model belongs to the class of potential game (Monderer and Shapley, 1996). Then, by invoking the properties of potential game, we analyze the stability of equilibria under a broad class of evolutionary dynamics. We also investigate the social optimal distribution of work start times.

1 There are some subsequent works that employ Henderson (1981)'s assumption, e.g., Mun and Yonekawa (2006) and Yoshimura and Okumura (2001), in the context of a model of flextime. Empirical support for this assumption is provided in Wilson (1988).
THE MODEL

Basic Assumptions

We consider a city consisting of a CBD and a residential area. The CBD and the residential area are connected by a single road which has a single bottleneck with capacity $\mu$. For simplicity, we assume that the bottleneck is represented by a point queue model, and that there are no travel costs other than a queuing time cost at the bottleneck. Thus, the departure time from the bottleneck is the arrival time at the CBD.

All of $N$ workers commute from the residential area to the CBD in which all production takes place, and work a workday of fixed length $H$. Firms in the CBD are perfectly competitive and produce homogeneous goods. Each firm chooses the work start time $t_i$ from discrete set $\{t_1, t_2, \cdots, t_T\}$ so as to maximize profit. We suppose that $t_i = t_{i-1} + \tau$ ($i = 2, 3, \cdots, T$) and that $t_T \leq t_1 + H$. For notational purpose, the firms starting work at $t_i$ are indexed by $i$.

Workers maximize their net income $u_i(t)$, which is defined as wage minus commuting cost:

$$\max_{t} u_i(t) = w_i - c_i(t),$$

where $t$ denotes the departure time from the bottleneck, and $w_i$ is the wage from firm $i$. $c_i(t)$ is the commuting cost, which is defined as the sum of queuing time and schedule delay costs:

$$c_i(t) = q(t) + s(t, t_i),$$

where $q(t)$ is the queuing time cost for a worker departing the bottleneck at $t$, and $s(t, t_i)$ denotes the schedule delay cost. The schedule delay cost $s(t, t_i)$ is supposed to be quadratic:

$$s(t, t_i) = \beta (t_i - t)^2.$$  

We assume that each firm hires a single worker and has an instantaneous production function at time $t$ of the form

$$\alpha N(t),$$

where $\alpha$ is a constant representing technology, and $N(t)$ is the total number of workers on duty in the CBD at time $t$. Denoting $N_i$ as the total number of workers employed by firm $i$, we can represent $N(t)$ as follows (Figure 1):

\[\text{Figure 1: The total number of employees on duty in the CBD}\]
\[ N(t) = \begin{cases} N_1 & \text{if } t \in (t_1, t_2), \\ N_1 + N_2 & \text{if } t \in [t_2, t_3), \\ \vdots & \\ \sum_{k=1}^l N_k & \text{if } t \in [t_j, t_{j+1}), \\ \vdots & \\ \sum_{k=2}^l N_k & \text{if } t \in [t_1 + H, t_2 + H), \\ \vdots & \\ \sum_{k=j}^l N_k & \text{if } t \in [t_{j-1} + H, t_j + H), \\ \vdots & \\ N_T & \text{if } t \in [t_{T-1} + H, t_T + H). \\ \end{cases} \]  

(5)

Output \( f_i \) of a firm \( i \) for one day is given by integrating the instantaneous production function from \( t_i \) to \( t_i + H \):

\[ f_i = \alpha \int_{t_i}^{t_i+H} N(t) dt = \alpha \left[ \sum_{k=1}^{t-1} (t_k + H - t_i) N_k + \sum_{k=2}^{i} \{ (t_k + H - (t_{k-1} + H)) \} \sum_{j=k} T N_j \right] \]

(6)

Hence the profit maximization of a firm is represented as

\[ \max_i \pi_i = f_i - w_i. \]  

(7)

**Equilibrium Conditions and their Equivalent Formulation**

An equilibrium of the model is defined as a state that satisfies the following seven conditions:

\[ n_i(t)[c_i^* - (q(t) + s(t, t_i))] = 0 \quad \forall i, t, \]  

(8)

\[ n_i(t) \geq 0, c_i^* - (q(t) + s(t, t_i)) \geq 0 \quad \forall i, t, \]  

(9)

\[ q(t)[\mu - \sum_k n_k(t)] = 0 \quad \forall t, \]  

\[ q(t) \geq 0, \mu - \sum_k n_k(t) \geq 0 \]  

(10)

\[ \int n_i(t) dt = N_i \quad \forall i, \]  

(11)

\[ N_i[u^* - (w_i - c_i^*)] = 0 \quad \forall i, \]  

(12)

\[ \sum_k n_k = N_i \]  

(13)

\[ M_i(f_i - w_i) = 0 \quad \forall i, \]  

\[ M_i \geq 0, f_i - w_i \geq 0 \quad \forall i, \]  

(14)

where \( n_i(t) \) is arrival rate (the number of arrivals at the CBD at each time) of workers employed by firms \( i \) at time \( t \), and \( M_i \) is the number of firm \( i \). \( c_i^* \) and \( u_i^* \) denote equilibrium commuting cost and net income of a worker starting work at \( t_i \), respectively.

The first condition (8) means that, at equilibrium, no worker can improve his/her own commuting cost by changing the CBD arrival time unilaterally. The second condition (9) requires that the total arrival rate \( \sum_k n_k(t) \) is equal to the bottleneck capacity \( \mu \) if there is a queue; otherwise, the total arrival rate is (weakly) lower than the bottleneck capacity. The third condition (10) represents flow conservation for commuting demand. The forth condition (11) implies that each worker has no incentive to change firm \( i \) under equilibrium. The fifth condition (12) is the conservation law of population. The
sixth condition (13) means that each firm has no incentive to change its work start time $t_i$ at equilibrium. The last condition (14) is labor market clearing condition.

$$\begin{align*}
\text{The equilibrium conditions (11), (13), and (14) can reduce to}
\begin{cases}
N_i [u^* - (f_i - c_i^*)] = 0 \\
N_i \geq 0, u^* - (f_i - c_i^*) \geq 0
\end{cases} \quad \forall i.
\end{align*}$$

Hence we can obtain equilibrium solutions from the conditions (8), (9), (10), (12), and (15).

In order to investigate uniqueness and stability of the equilibrium in the following sections, we convert the equilibrium conditions into the optimization problem.

**Lemma 1** \( ([n_i(t)], [N_i]) \) is an equilibrium if and only if it satisfies the Kuhn-Tucker conditions for the following optimization problem:

$$\begin{align*}
\max_{\{n_i(t), [N_i]\}} P &= \frac{1}{2} \sum_k N_k f_k - \sum_k \int n_k(t) s(t, t_k) \, dt, \\
\text{s.t., } \sum_k n_k(t) &\leq \mu, \quad \int n_i(t) \, dt = N_i, \quad \sum_k N_k = N, \\
n_i(t) \geq 0, &\quad N_i \geq 0,
\end{align*}$$

where \( f_i \) and \( s(t, t_k) \) are obtained from (6) and (3), respectively.

It is worth noting that Lemma 1 implies that our model belongs to the class of potential game (Monderer and Shapley, 1996), and that we are now ready to examine uniqueness and stability of the equilibrium.

**EQUILIBRIUM PATTERNS OF WORK START TIMES AND BOTTLENECK CONGESTION**

**Uniqueness of Equilibrium**

We examine uniqueness and stability of the equilibrium of the model. In order to simplify the examination, we consider the case that \( T = 2 \) in this section. In this case, the equilibrium patterns of work start times and bottleneck congestion can be classified as follows:

**Pattern 1:** Work start times of firms are concentrated (Figure 2a).

**Pattern 2.1:** Work start times of firms are staggered, and the rush hour is divided into two time intervals (Figure 2b).

**Pattern 2.2:** Work start times of firms are staggered, and the rush hour is treated as a single period (Figure 2c).

We first investigate the uniqueness of the equilibrium by checking the concavity of the objective function \( P \) of (16). Noting that the rush hour is divided only if \( 2 \mu t - N > 0 \), we can rewrite the optimization problem (16) as follows:

$$\begin{align*}
\max_{N_1} P(N_1) &= P_1(N_1) - P_2(N_2), \\
\text{s.t., } 0 \leq N_1 \leq N.
\end{align*}$$
The first term $P_1(N_1)$ and the second term $P_2(N_1)$ of the objective function $P(N_1)$ are given by

$$P_1(N_1) = -2\alpha \tau N_1 (N - N_1),$$

and

$$P_2(N_1) = \begin{cases} \frac{\beta}{12\mu^2} \left( N^3 - 3NN_1(N - N_1) \right) & \text{if } 2\mu \tau > N, \\ \frac{\beta}{12\mu^2} \left( N^3 - 12\mu \left( 1 - \frac{\tau \mu}{N} \right) N_1(N - N_1) \right) & \text{if } 2\mu \tau \leq N. \end{cases}$$

Examining the signs of $d^2P_1(N_1)/dN_1^2$ and $d^2P_2(N_1)/dN_1^2$, we have the following lemma:

**Lemma 2** $P_1(N_1)$ and $P_2(N_1)$ are strictly convex functions of $N_1$.

Lemma 2 implies that $P(N_1)$ is not necessarily concave function since the first term $P_1(N_2)$ of $P(N_1)$ is strictly convex function. Therefore we obtain the following proposition:

**Proposition 1** The equilibrium of the model is generally non-unique.

**Stability of Equilibria**

Because our model generally has multiple equilibria, we next examine their stability. As is well known in evolutionary game theory (e.g., Weibull, 1995; Fudenberg and Levine, 1998; Sandholm, 2010), local maximizers of the objective function $P$ are all locally stable equilibrium states under a broad class of evolutionary dynamics (for details, see, e.g., Sandholm (2001, 2005), Oyama (2009a, b), Fujishima (2012)). Therefore we can assess the stability of the equilibrium $N_1^*$ by checking the sign of $d^2P(N_1^*)/dN_1^2$:

$$d^2P(N_1^*)/dN_1^2 = \begin{cases} 2 \left( \frac{2\alpha - \beta N}{4\mu^2} \right) & \text{if } 2\mu \tau > N, \\ 2\tau \left( \frac{2\alpha - \beta}{\mu} \left( 1 - \frac{\tau \mu}{N} \right) \right) & \text{if } 2\mu \tau \leq N. \end{cases}$$

![Figure 2: Equilibrium patterns of work start times and bottleneck congestion](image)
This leads to the following proposition:

**Proposition 2** In the case that each firm chooses the work start time from two alternatives,

1. Pattern 1 is a stable equilibrium if and only if the parameters $\alpha, \beta, \mu, \tau, N$ satisfy
   
   \[
   2\alpha \tau > \frac{\beta N}{4\mu^2} \quad \text{and} \quad 2\alpha > \frac{\beta}{\mu} \left(1 - \frac{\tau \mu}{N}\right). \tag{21}
   \]

2. Pattern 2.1 is a stable equilibrium if and only if the parameters $\alpha, \beta, \mu, \tau, N$ satisfy
   
   \[
   2\alpha \tau < \frac{\beta N}{4\mu^2} \quad \text{and} \quad 2\mu \tau > N. \tag{22}
   \]

3. Pattern 2.2 is a stable equilibrium if and only if the parameters $\alpha, \beta, \mu, \tau, N$ satisfy
   
   \[
   2\alpha < \frac{\beta}{\mu} \left(1 - \frac{\tau \mu}{N}\right) \quad \text{and} \quad 2\mu \tau \leq N. \tag{23}
   \]

Figure 3 illustrates the relation between stable equilibria and parameters $\alpha, \beta, N, \mu$ and $\tau$.

**SOCIAL OPTIMAL**

In order to compare the equilibrium and social optimal of the model, we consider the following optimization problem:

\[
\max_{\{n_i(t)\}, \{N_k\}} SW = \sum_k N_k f_k(t) - \sum_k \int n_k(t)(q(t) + s(t, t_k)) \, dt,
\]

\[
s. \ t., \ \sum_k n_k(t) \leq \mu, \quad \int n_i(t) \, dt = N_i, \quad \sum_k N_k = N,
\]

\[
n_i(t) \geq 0, \quad N_i \geq 0,
\]

This is the problem of finding a state that maximizes the social welfare in the model, subject to the physical constraints.
Comparing between the optimization problems (16) and (24), we can find the difference, 
\[ SW - P = \frac{1}{2} \sum_k N_k w_k - \sum_k \int \sum_n \int n_k(t) q(t) dt, \]
between the objective functions. This implies that work start times of social optimal can be more concentrated than those of the equilibrium, and therefore any staggering of work hours may be harmful.

CONCLUSION

In this study, we examine the equilibrium patterns of work start times and bottleneck congestion by extending the Henderson (1981)'s model to incorporate the bottleneck congestion. The main results are summarized as follows: 1) the extended model belongs to the class of potential game, 2) equilibrium of the model is generally non-unique, 3) in the case that each firm chooses the work start time from two alternatives, the stable equilibrium patterns of work start times and bottleneck congestion is uniquely determined. It should be noted that a number of assumptions are introduced to simplify the analysis, e.g., the number of work start times that firms can choose is two. In order to investigate the robustness of our findings, we need to analyze the model with multiple work start times. It is also valuable for future research to show the tax policy for achieving social optimal.

REFERENCES